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LETTER TO THE EDITOR

A new technique for deriving the large-N solution of the Klein–Gordon equation

S S Stepanov and R S Tutik

Department of Physics, Dniepropetrovsk State University, Dniepropetrovsk, SU-320625, Ukraine, USSR

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Abstract. A new procedure for deriving the energy values of the Klein–Gordon equation with a potential having both scalar and vector components is described. Using the $\hbar$-expansion and the quantization condition recursion formulae were obtained which can readily be applied to any $1/N$-expansion scheme. These formulae have simple structures both for background and excited states and provide, in principle, the calculation of $1/N$-corrections up to an arbitrary order. Contrary to the methods elaborated formerly our technique does not involve converting the initial equation to the shortened, Schrödinger-like form. For the Coulomb potential the solutions of the Klein–Gordon and shortened equations are compared.

The method of large-N expansion, where N is the number of spatial dimensions, has emerged in recent years as a very useful approximation scheme in non-relativistic quantum mechanics (for a review see e.g. Chatterjee (1990)). Several attempts have been made to extend this method to the relativistic case within the Klein–Gordon equation. In particular, Nieto (1979), Miramontes and Pajares (1984) have studied the large-N limit for the Coulomb potential. For a general spherically symmetric potential an extension of the unshifted $1/N$-expansion (Mlodinov and Shatz 1984) has been made by Chatterjee (1986), while the shifted large-N technique (Sukhatme and Imbo 1983) has been adopted by Panja and Dutt (1988), Panja *et al* (1989).

The characteristic feature of these approaches is the conversion of the Klein–Gordon equation into a Schrödinger-like equation after neglecting some terms. Then one may apply any standard $1/N$-expansion formalism in a straightforward manner.

But, in the case of excited states, the standard methods of $1/N$-expansion become extremely cumbersome. Moreover, the above extensions were done only with a Lorentz vector potential.

In this connection we adapt an $\hbar$-expansion technique for obtaining the energy eigenvalues of a scalar particle in the presence of both a fourth-component Lorentz vector potential and a scalar one. Without neglecting any terms of the equation, it enables us to find the $1/N$-corrections of an arbitrary order. Derived recursion formulae have a simple structure not only in the case of background states but for the excited states as well.

The proposed technique is a subsequent development of the $\hbar$-expansion, applied recently for the (l, E)-plane analysis of the bound-state problem in the non-relativistic case (Kobylinsky *et al* 1990a, b, Korobov *et al* 1990) and within the Klein–Gordon equation (Kobylinsky *et al* 1990c).

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Method. The reduced radial part of the N-dimensional Klein-Gordon equation for a scalar particle in the presence of a fourth-component Lorentz vector potential $V(r)$ and a scalar potential, included in the mass term $m(r)$ following the dynamical mass model (Gunion and Li 1975), is given by

$$\hbar^2 U''(r) = \left(m^2(r)c^2 - \frac{1}{c^2}[E - V(r)]^2 + \frac{1}{4r^2}[\bar{k} + \hbar(a-1)][\bar{k} + \hbar(a-3)] \right) U(r) \tag{1}$$

where $\bar{k} = \hbar(2l + N - a)$, and a is some shift parameter (Sukhatme and Imbo 1983).

Usually, the authors proceed with solving equation (1) in terms of the expansion parameter $1/\bar{k}$. But actually this parameter is $\hbar/\bar{k}$. Therefore we can choose the alternative possibility of solving this equation by a series expansion in Planck's constant.

Substituting a logarithmic derivative, $C(r) = \hbar U'(r)/U(r)$, we rewrite equation (1) in the Riccati form

$$\begin{aligned} \hbar C^1(r) + C^2(r) &= (\bar{k}/2r)^2 + m^2(r)c^2 - (E - V(r))^2/c^2 \\ &\quad + \hbar\bar{k}(a-2)/2r^2 + \hbar^2(a-1)(a-3)/4r^2 \end{aligned} \tag{2}$$

which is the starting point in the method discussed.

Through the use of the $\hbar$-expansion

$$C(r) = \sum_{k=0}^{\infty} C_k(r)\hbar^k \tag{3}$$

$$E = \sum_{k=0}^{\infty} E_k \hbar^k \tag{4}$$

on comparing coefficients of various powers in $\hbar$, from equation (2) we obtain

$$C_0^2(r) = (\bar{k}/2r)^2 + m^2(r)c^2 - [E_0 - V(r)]^2/c^2 \tag{5}$$

$$C_1(r) = \frac{1}{2C_0(r)}\left[\gamma_1 \left(\frac{r_0}{r}\right)^2 + 2E_1 V(r)/c^2 - 2E_0 E_1/c^2 - C_0^1(r) \right] \tag{6}$$

$$\begin{aligned} C_k(r) = \frac{1}{2C_0(r)}\Bigg[& \gamma_k \left(\frac{r_0}{r}\right)^2 + 2E_k V(r)/c^2 - \frac{1}{c^2}\sum_{i=0}^{k} E_i E_{k-i} \\ & - C_{k-1}^1(r) - \sum_{i=1}^{k-1} C_i(r) C_{k-i}(r) \Bigg] \end{aligned} \tag{7}$$

where $\gamma_1 = \bar{k}(a-2)/2r_0^2$, $\gamma_2 = (a-1)(a-3)/4r_0^2$, $\gamma_3 = \gamma_4 = \ldots = 0$.

This system is solvable if we know $C_0(r)$, which includes the quantity E_0. In the classical limit, $\hbar = 0$, the Klein-Gordon particle executes the motion along a stable circular orbit with the energy

$$E_0 = V_{\text{eff}}(r_0) = V(r_0) + m(r_0)c^2\left(1 + \frac{\bar{k}^2}{4r_0^2 m^2(r_0)c^2}\right)^{1/2}. \tag{8}$$

The radius r_0, of this orbit is the location of the minimum of the effective potential. It is determined as the positive root of the equation

$$r_0^3 m^1(r_0)c^2 + r_0^3 V^1(r_0)\left[1 + \frac{\bar{k}^2}{4r_0^2 m^2(r_0)c^2}\right]^{1/2} = \frac{\bar{k}^2}{4m(r_0)}. \tag{9}$$

Substituting E_0, $C_0^1(r_0)$ and $C_0(r_0)=0$ into equation (6) we compute E_1. Next we may find the quantity $C_1(r)$ and so on.

But for the excited states one needs taking into account zeros of the radial wavefunctions. The $\hbar$-expansion simplifies solving this problem by permitting us to use a quantization condition.

In the case of the nth radially excited state the wavefunction $U(r)$ has n real nodes, so its logarithmic derivative, $C(r)$, has n simple poles at these points. There results in the quantization condition (Zwaan 1929, Dunham 1932)

$$\frac{1}{2\pi i}\oint C(r)\,dr = n\hbar \qquad n=0,1,2,\ldots. \tag{10}$$

This condition is exact and is widely used for deriving the higher-order corrections to the WKB approximation. But in the WKB approach a classical limit is carried out with the rule

$$\hbar\to 0 \qquad n\to\infty \qquad \hbar n = \text{constant}. \tag{11}$$

Our method, being complementary to the WKB one, involves the alternative possibility

$$\hbar\to 0 \qquad n=\text{constant}\ \hbar n\to 0. \tag{12}$$

Hence, making use of expansion (3) it follows that

$$\frac{1}{2\pi i}\oint C_1(r)\,dr = n \tag{13}$$

$$\frac{1}{2\pi i}\oint C_k(r)\,dr = 0 \qquad k\neq 1. \tag{14}$$

To illustrate the application of these conditions, we consider the derivation of the energy eigenvalues.

Recurrence formulae. We now put $x=(r-r_0)/r_0$ and put the expression for $C_0(x)$ into another form by means of a series expansion about the point $x=0$

$$C_0(x) = -\omega x(1+a_1x+a_2x^2+\ldots)^{1/2} \tag{15}$$

where the minus sign is chosen from a boundary condition, and

$$\omega^2 = 3\bar{k}^2/4r_0^2 + 2E_0V_2/c^2 + c^2(2m_0m_2+m_1^2) - \frac{1}{c^2}(2V_0V_2+V_1^2) \tag{16}$$

$$a_\alpha = \frac{1}{\omega^2}\left[(-1)^\alpha\frac{3+\alpha}{4r_0^2}\bar{k}^2 + 2E_0V_{2+\alpha}/c^2 + c^2\sum_{j=0}^{\alpha+2} m_jm_{\alpha+2-j} - \frac{1}{c^2}\sum_{j=0}^{\alpha+2} V_jV_{\alpha+2-j}\right]. \tag{17}$$

Here the quantities $V_\alpha = r_0^\alpha V^{(\alpha)}(r_0)/\alpha!$ and $m_\alpha = r_0^\alpha m^{(\alpha)}(r_0)/\alpha!$ are the coefficients of the Taylor-series expansions

$$V(r) = \sum_{\alpha=0}^{\infty} V_\alpha x^\alpha \qquad m(r) = \sum_{\alpha=0}^{\infty} m_\alpha x^\alpha. \tag{18}$$

From equation (15) it is clear that $x=0$ is the simple zero for the function $C(x)$. Consequently, $C_1(x)$ has the simple pole at this point, while $C_k(x)$ has the pole of the order of $(2k-1)$. We then conclude that $C_k(x)$ can be expanded in a Laurent series

$$C_k(x) = x^{1-2k}\sum_{\alpha=0}^{\infty} C_\alpha^k x^\alpha. \tag{19}$$

Now, with definition of the residues, we can also express the quantization conditions (13) and (14) in terms of the coefficients C_α^k as

$$\operatorname{res} C_1(0) = C_0^1 = n/r_0 \tag{20}$$

$$\operatorname{res} C_k(0) = C_{2k-2}^k = 0 \qquad k \neq 1. \tag{21}$$

Unifying our notation, we rewrite the expansion (15) in the form

$$C_0(x) = x \sum_{\alpha=0}^{\infty} C_\alpha^0 x^\alpha. \tag{22}$$

The coefficients C_α^0 are related to a_α by the equations

$$\begin{aligned} &C_0^0 = -\omega \qquad C_1^0 = -\frac{\omega a_1}{2} \\ &C_\alpha^0 = \frac{1}{2\omega}\left(\sum_{j=1}^{\alpha-1} C_j^0 C_{\alpha-j}^0 - \omega^2 a_\alpha\right). \end{aligned} \tag{23}$$

Thus, the formulae (20), (21) and (23) had already defined several coefficients C_α^k. For the remaining ones the substitution of the series (19) and (22) into equation (7) yields, letting $\alpha \neq 2k-2$,

$$C_\alpha^k = \frac{1}{2C_0^0}\left[\left((-1)^\alpha \gamma_k(\alpha-2k+3) + \frac{2E_k}{c^2} V_{\alpha-2k+2}\right)\theta(\alpha-2k+2) - \frac{3-2k+\alpha}{r_0} C_\alpha^{k-1} - \sum_{j=1}^{k-1}\sum_{\beta=0}^{\alpha} C_\beta^j C_{\alpha-\beta}^{k-j} - 2\sum_{\beta=1}^{\alpha} C_\beta^0 C_{\alpha-\beta}^k\right] \tag{24}$$

where we use the step function

$$\Theta(\alpha) = \begin{cases} 1 & \alpha \geqslant 0 \\ 0 & \alpha < 0 \end{cases}. \tag{25}$$

If we let $\alpha = 2k-2$, making use of equation (7) and the quantization relations (20) and (21), we would derive the recurrence formulae for the energy eigenvalues. Indeed, if we take $k=1$, there results

$$C_1(x)|_{x\to 0} = -\frac{1}{2\omega x}\left[\gamma_1 + \frac{2}{c^2} E_1(V_0 - E_0) + \frac{\omega}{r_0}\right]. \tag{26}$$

But due to (18), res $C_1(0)$ becomes n/r_0 which gives immediately

$$E_1 = \frac{c^2}{2(E_0 - V_0)}\left(\gamma_1 + \frac{(2n+1)\omega}{r_0}\right). \tag{27}$$

On applying the condition (21), when $k \neq 1$, we have

$$E_k = \frac{c^2}{2(E_0 - V_0)}\left(\gamma_k - \frac{1}{r_0} C_{2k-2}^{k-1} - \frac{1}{c^2}\sum_{j=1}^{k-1} E_j E_{k-j} - \sum_{j=1}^{k-1}\sum_{\beta=0}^{2k-2} C_\beta^j C_{2k-2-\beta}^{k-j} - 2\sum_{\beta=1}^{2k-2} C_\beta^0 C_{2k-2-\beta}^k\right) \tag{28}$$

where the coefficients C_β^j had been defined earlier.

Finally notice, that the coefficients of the $1/N$-expansion, $E = \mathscr{E}^{(0)} + \mathscr{E}^{(1)}/\bar{k} + \mathscr{E}^{(2)}/\bar{k}^2 + \ldots$, may be deduced from the $\hbar$-expansion coefficients through the relation

$$\mathscr{E}^{(k)} = (\hbar\bar{k})^k E_k. \tag{29}$$

Application to the Coulomb problem. The method outlined above we apply to the case of an attractive Coulomb potential $V(r) = -\beta/r$, $\beta = ze^2$. Putting $m(r) = m = 1$, the solution of equation (9), when expanded in powers of $1/c^2$, becomes

$$r_0 = \frac{\bar{k}^2}{4\beta}\left(1-\frac{4\beta^2}{\bar{k}^2c^2}\right)^{1/2} = \frac{\bar{k}^2}{4\beta}\left[1-2\left(\frac{\beta}{\bar{k}c}\right)^2-2\left(\frac{\beta}{\bar{k}c}\right)^4-4\left(\frac{\beta}{\bar{k}c}\right)^6+\ldots\right]. \tag{30}$$

Taking into account that $E_0 = (4\beta c^2/\bar{k}^2)r_0$, $\omega = 2\beta/\bar{k}$ and $V_k = (-1)^{k+1}\beta/r_0$ we have

$$E_1 = \frac{4\beta^2}{\bar{k}^3}(a-2+p)+\frac{8\beta^4(a-2)}{\bar{k}^5c^2}+\frac{24\beta^6(a-2)}{\bar{k}^7c^4}+\frac{80\beta^8(a-2)}{\bar{k}^9c^6}+\ldots \tag{31}$$

where $p = 2n+1$.

Following Panja and Dutt (1988), we now choose the shift parameter a in such a manner that the non-relativistic part of expression (31) vanishes. This gives $a = 2-p$ and we obtain

$$E_1 = -\frac{8\beta^4 p}{\bar{k}^5c^2}-\frac{24\beta^6 p}{\bar{k}^7c^4}-\frac{80\beta^8 p}{\bar{k}^9c^6}-\ldots \tag{32}$$

$$E_2 = -\frac{8\beta^4 p^2}{\bar{k}^6c^2}-\frac{72\beta^6 p^2}{\bar{k}^8c^4}-\frac{336\beta^8 p^2}{\bar{k}^{10}c^6}-\ldots. \tag{33}$$

Collecting terms of the same order in $1/c^2$, there results

$$\begin{aligned} E = c^2 &- \frac{2\beta^2}{\bar{k}^2}-\frac{\beta^4}{c^2\bar{k}^4}\left[2+8\left(\frac{p\hbar}{\bar{k}}\right)+8\left(\frac{p\hbar}{\bar{k}}\right)^2+\ldots\right] \\ &-\frac{\beta^6}{c^4\bar{k}^6}\left[4+24\left(\frac{p\hbar}{\bar{k}}\right)+72\left(\frac{p\hbar}{\bar{k}}\right)^2+\ldots\right] \\ &-\frac{\beta^8}{c^6\bar{k}^8}\left[10+80\left(\frac{p\hbar}{\bar{k}}\right)+336\left(\frac{p\hbar}{\bar{k}}\right)^2+\ldots\right]-\ldots. \end{aligned} \tag{34}$$

Our analytic result (34) reproduces the exact one (Nieto 1979)

$$E = c^2\left[1+\frac{4\beta^2}{\hbar^2c^2}\left(2n+1+\sqrt{(2l+1)^2-\frac{4\beta^2}{\hbar^2c^2}}\right)^{-2}\right]^{-1/2} \tag{35}$$

if we expand it in powers of $\hbar$ and $1/c^2$.

It should be emphasized that the approximations derived by Chatterjee (1986) and Panja *et al* (1988, 1989) may be easily restored from our formulae. The reducing of the Klein-Gordon equation to the Schrödinger-like form consists in neglecting the term $2(E-E_0)(V(r)-V(r_0))$ in the identity

$$\begin{aligned} (E-V(r))^2 = (E_0-V(r))^2 &+ (E-V(r_0))^2-(E_0-V(r_0))^2 \\ &-2(E-E_0)(V(r)-V(r_0)). \end{aligned} \tag{36}$$

Then, within our procedure, we have to drop out only the quantities $2E_kV(r)/c^2$ in equation (7) and $2E_kV_{\alpha-2k+2}/c^2$ in the recurrence formula (24). This results in replacing the term $336(p\hbar/\bar{k})^2$ by $464(p\hbar/\bar{k})^2$ in expansion (34). Of course, the change in coefficients will appear in the terms of higher order in $\hbar$ and also $1/c^2$. From equation (34) it is clear that such shortening of the Klein-Gordon equation almost does not change its solution in the 'quasi-non-relativistic' case, when β is small. The marked difference between these solutions becomes apparent only when relativistic corrections become significant. The results of calculation, demonstrated in table 1, confirm this inference.

Table 1. The binding energies of $n=0$, $l=3$ states for a spin-zero particle in the Coulomb potential $V(r)=-\beta/r$ within both the Schrödinger-like and the Klein–Gordon equations, computed with various $1/N$-corrections ($E^{(m)}=E_0+E_1+\ldots+E_m$). The deviation from the exact values, defined by (35), is given in the percentage error. The quantity in parenthesis denotes the order of values, e.g., $(-6)=10^{-6}$.

		The Schrödinger-like equation		The Klein–Gordon equation	
	m	$E^{(m)}$	% error	$E^{(m)}$	% error
$\beta=\frac{1}{2}$	1	0.992 141 3017	2.29 (−4)	0.992 141 3017	2.29 (−4)
	2	0.992 139 3256	2.93 (−5)	0.992 139 3260	2.94 (−5)
	3	0.992 139 0717	3.73 (−6)	0.992 139 0723	3.79 (−6)
	4	0.992 139 0389	4.28 (−7)	0.992 139 0396	4.91 (−7)
	5	0.992 139 0347	3.49 (−10)	0.992 139 0353	6.37 (−8)
	6	0.992 139 0341	5.51 (−8)	0.992 139 0348	8.31 (−9)
$\beta=3$	1	0.625 447 8197	2.72	0.625 447 8197	2.72
	2	0.611 255 1993	3.88 (−1)	0.614 620 9935	9.41 (−1)
	3	0.605 324 2959	5.86 (−1)	0.611 008 3753	3.47 (−1)
	4	0.602 757 6548	1.01	0.609 711 8950	1.34 (−1)
	5	0.601 621 0519	1.19	0.609 221 5190	5.38 (−2)
	6	0.601 109 4475	1.28	0.609 028 6241	2.21 (−2)

To conclude, we have developed a simple recursion formalism for any $1/N$-expansion scheme of the Klein–Gordon equation with potentials having both vector and scalar components and this formalism provides, in principle, the calculation of $1/N$-corrections up to an arbitrary order both for background and excited states.

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